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A note on the orientation of symmetric rigid bodies sedimenting in a power-law fluid

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Abstract

We study the terminal orientation of symmetric bodies translating in a quiescent liquid modeled by the power-law fluid. We are able to show by invoking the symmetries of the sedimenting body and the Stokes flow field that at small Reynolds numbers, the competition of inertial and shear-thinning (or shear-thickening) contributions to the torque does not cause the tilt angle that is observed in experiments performed on viscoelastic liquids with shear-thinning properties.

Keywords: Sedimentation; Orientation; Power-law fluid

1. Introduction

It is a well established fact that homogeneous bodies of revolution around an axis, a, with fore–aft symmetry, when dropped in a quiescent liquid, will orient themselves with respect to the direction of gravity ($\mathbf{g}$) depending upon their shape and upon the nature of the fluid in which they are immersed. If, for example, we are considering a prolate spheroidal object falling in a Newtonian fluid such as water, then the body falls with a eventually becoming perpendicular to the direction of $\mathbf{g}$. However if the same body falls in a viscoelastic fluid where the inertial effects can be disregarded then a will eventually become parallel to $\mathbf{g}$ (see Fig. 1). Furthermore, it has also been observed that elongated bodies falling

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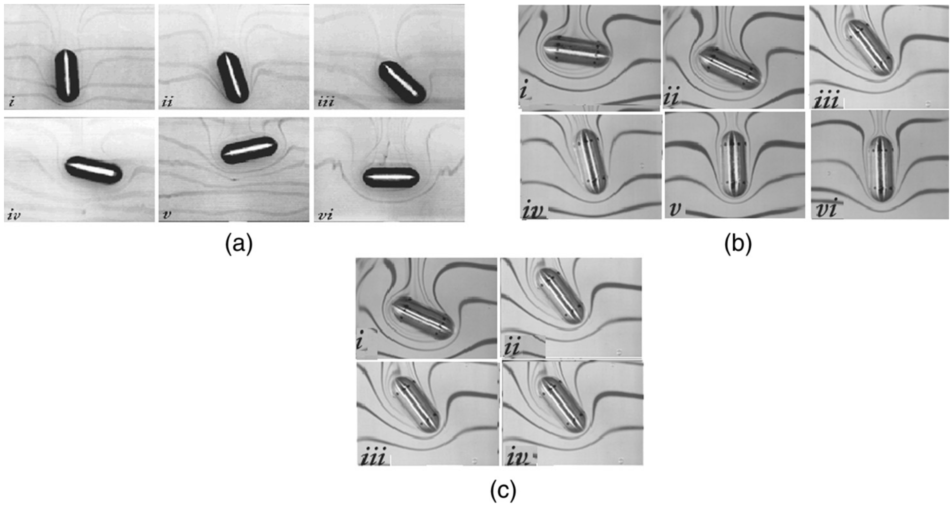

Fig. 1. Experimental observations on the orientation of a cylinder in (a) Newtonian liquid and (b) viscoelastic liquid. Fig. (c) shows the tilt angle phenomenon (Courtesy of D.D. Joseph).

in fluids with certain polymeric concentrations can take on angles between the horizontal and vertical orientations. These intermediate angles are referred to in the literature as *tilt angles* [1–4]. Experimental evidence for these phenomena are plentiful in the literature (see papers cited above and references cited therein); however a complete theoretical explanation is still not forthcoming.

Theoretical explanations of these observations are based on the heuristic proposition of Joseph and Feng [5] that the terminal angle is determined by the dominant inertial or viscoelastic torque imposed on the body by the fluid. It has been established [6–9] that though the competing torque theory of Joseph and Feng explains the terminal orientations of rigid bodies in Newtonian and purely viscoelastic liquids modeled by the second-order fluid equations, it is insufficient to explain the tilt angle. Yet another mechanism that may influence the orientation of bodies is shear-thinning [9,10]. In fact, numerical studies in two dimensions [10] have shown that a tilt angle is apparent in case of a shear-thinning liquid modeled by an Oldroyd-B fluid with the viscosity given by the Carreau–Bird viscosity law.

In this work, our aim is to consider the shear-thinning property in isolation. Since inertia and normal stresses apparently do not contribute to the tilt, we want to see whether shear-thinning/thickening by itself can result in the tilt angle. For this purpose we model our fluid with the power-law fluid equations whose constitutive equation is given by [11,12]

$$\mathbf{T}(\mathbf{u}, p) = -p\mathbf{I} + \eta(II_D)\mathbf{D}(\mathbf{u})$$

where the viscosity is given by

$$\eta(II_D) = \eta_1[\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u})]^{\frac{n-1}{2}}$$

with $\mathbf{D}(\mathbf{u}) = \frac{1}{2}[\text{grad}(\mathbf{u}) + \text{grad}^T(\mathbf{u})]$ and η_1 is a parameter with dimensions Pa s^n [11]. The power $\frac{n-1}{2}$ determines whether the viscosity of the fluid will increase (shear-thickening) or decrease (shear-thinning) with the shear rate. Therefore for a shear-thinning fluid, $n < 1$, for a Newtonian fluid, $n = 1$, and for a shear-thickening fluid, $n > 1$. In this work we show my means of a simple parity argument that shear-thinning or shear-thickening, by itself, adds no contribution to the torque on the body at low

Reynolds numbers. This seems to suggest that the tilt angle is therefore a result of the combined effect of inertia, viscoelasticity and shear-thinning properties of a liquid. The significance of this note is that it is a positive step in the systematic study of the orientation phenomenon. It allows us to eliminate certain fluid models for studying this phenomenon and suggests other appropriate possible mechanisms that might better explain the tilt angle.

2. Calculation of the torque

We assume that the body $\mathcal{B}$ is free-falling in an unbounded power-law fluid, $\mathcal{F}$, under the influence of gravity g with a translational velocity of ξ. The problem will be studied in a frame which is attached to the body so that the motion of $\mathcal{B}$ when observed from the attached frame F is steady [6,7,9,13–15]. In this section, we will find an expression for the torque, $\mathcal{M} = \mathcal{M}_i \mathbf{e}_i$ (for $i = 1, 2, 3$), imposed by $\mathcal{F}$ on $\mathcal{B}$.

The equations of motion in non-dimensional form are given by

$$\begin{aligned} &\mathrm{Re}\,\mathbf{u} \cdot \mathrm{grad}(\mathbf{u}) = \mathrm{div}\,\mathbf{T}(\mathbf{u}, p) \\ &\mathrm{div}\,(\mathbf{u}) = 0 \\ &\mathbf{u} = 0 \text{ at } \Sigma \equiv \partial\Omega \\ &\lim_{|\mathbf{x}|\to\infty} (\mathbf{u}(\mathbf{x}) + \xi) = 0 \end{aligned} \tag{1}$$

where ξ is a vector, with components (ξ_1, ξ_2, ξ_3). Here, $\mathbf{T}(\mathbf{u}, p)$ is given by

$$\mathbf{T}(\mathbf{u}, p) = -p\mathbf{I} + \lambda[\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u})]^{\frac{n-1}{2}}\mathbf{D}(\mathbf{u}) \tag{2}$$

$$= \mathbf{T}_P(p) + \lambda\mathbf{T}_{PL}(\mathbf{u}) \tag{3}$$

where λ above represents a non-dimensional parameter, related to η_1, that characterizes the shear-thinning nature of the liquid. Following (see [7,9]) and defining the tensor $\mathbf{T}_N(\mathbf{u}, p) = -p\mathbf{I} + 2\mathbf{D}(\mathbf{u})$, we introduce the auxiliary fields $(\mathbf{h}^{(i)}, p^{(i)})$, $(\mathbf{h}^{(i)}, P^{(i)})$ which satisfy the equations

$$\begin{aligned} &\mathrm{div}\,\mathbf{T}_N(\mathbf{h}^{(i)}, p^{(i)}) = 0 \\ &\mathrm{div}\,\mathbf{h}^{(i)} = 0 \\ &\lim_{|\mathbf{x}|\to\infty} \mathbf{h}^{(i)}(\mathbf{x}) = 0 \\ &\mathbf{h}^{(i)}(\mathbf{y}) = \mathbf{e}_i, \qquad \mathbf{y} \in \Sigma, \end{aligned} \tag{4}$$

$$\begin{aligned} &\mathrm{div}\,\mathbf{T}_N(\mathbf{H}^{(i)}, P^{(i)}) = 0 \\ &\mathrm{div}\,\mathbf{H}^{(i)} = 0 \\ &\lim_{|\mathbf{x}|\to\infty} \mathbf{H}^{(i)}(\mathbf{x}) = 0 \\ &\mathbf{H}^{(i)}(\mathbf{y}) = \mathbf{e}_i \times \mathbf{y}, \qquad \mathbf{y} \in \Sigma. \end{aligned} \tag{5}$$

Multiplying Eqs. (1) by $\mathbf{H}^{(i)}$ and integrating by parts over Ω, we have, upon some rearrangement, the expression

$$\int_\Sigma (\mathbf{e}_i \times \mathbf{x}) \cdot \mathbf{T}(\mathbf{u}) \cdot \mathbf{n} = \mathrm{Re} \int_\Omega \mathbf{u} \cdot \mathrm{grad}(\mathbf{u}) \cdot \mathbf{H}^{(i)} + \lambda \int_\Omega \mathbf{T}_{PL}(\mathbf{u}) : \mathbf{D}(H^{(i)}). \tag{6}$$

We observe that the term on the left hand side of Eq. (6) is in fact

$$\mathbf{e}_i \cdot \int_{\Sigma} \mathbf{x} \times \cdot \mathbf{T}(\mathbf{u}, p) \cdot \mathbf{n} = -\mathcal{M}, \tag{7}$$

where $\mathcal{M}$ is the net torque.

Hence, we have, in short

$$\mathcal{M} = \mathrm{Re}\,\mathcal{M}^I(\mathbf{u}) + \lambda \mathcal{M}^{PL}(\mathbf{u})$$

which are the inertial and shear-thinning components of the net torque, respectively. In order to represent the torques at first order in Re, we write $\mathbf{u} = \mathbf{u}_S + \mathbf{w}$ [6,7,13] where $\mathbf{u}_S$ represents the velocity field corresponding to the Stokes problem (i.e. the equation for $Re = 0$). As a result, torque $\mathcal{M}$ becomes

$$\mathcal{M} = \mathrm{Re}\,\mathcal{M}^I(\mathbf{u}_S) + \lambda \mathcal{M}^{PL}(\mathbf{u}_S) + \mathcal{N}(\mathbf{w}) \tag{8}$$

where

$$\mathcal{M}_i^I(\mathbf{u}_S) = -\int_{\Omega} \mathbf{u}_S \cdot \mathrm{grad}(\mathbf{u}_S) \cdot \mathbf{H}^{(i)} \tag{9}$$

$$\mathcal{M}_i^{PL}(\mathbf{u}_S) = -\int_{\Omega} [\mathbf{D}(\mathbf{u}_S) : \mathbf{D}(\mathbf{u}_S)]^{\frac{n-1}{2}} \mathbf{D}(\mathbf{u}_S) : \mathbf{D}(\mathbf{H}^{(i)}). \tag{10}$$

Here $\mathcal{N}(\mathbf{w})$ represents higher order terms in Reynolds numbers. Our treatment is restricted to a first-order effect in *Re*. Therefore we effectively ignore the term $\mathcal{N}$.

In order to evaluate the torque, it is perhaps best to make use of the symmetries of the sedimenting body, $\mathcal{B}$. In this work, we take $\mathcal{B}$ to be a body with fore–aft symmetry, implying that it has three planes of reflection symmetry and one axis of rotational symmetry. This motivates the definitions below.

Definition 1 (*Rotational Symmetry*). We say that a body $\mathcal{B}$ has rotational symmetry about an axis, say x_1, if and only if

$$(x_1, x_2, x_3) \in \Sigma \Rightarrow (x_1, -x_2, x_3), (x_1, x_2, -x_3) \in \Sigma.$$

Definition 2 (*Symmetry Operators*). We define certain new symmetry classes next. We define the operators $\mathcal{P}_i$, $i = 1, 2, 3, 4$ such that

$$\mathcal{P}_1 f(x_1, x_2, x_3) := f(-x_1, x_2, x_3), \qquad \mathcal{P}_2 f(x_1, x_2, x_3) := f(x_1, -x_2, x_3)$$
$$\mathcal{P}_3 f(x_1, x_2, x_3) := f(x_1, x_2, -x_3), \qquad \mathcal{P}_4 f(x_1, x_2, x_3) := f(-x_1, -x_2, x_3).$$

Definition 3 (*Symmetry Class for Scalar Functions*). Suppose $\phi = \phi(x_1, x_2, x_3)$ is a scalar field. Then, we define the following symmetry classes:

$$\mathcal{C}_1^s := \{\phi : \mathcal{P}_4\phi = \phi, \mathcal{P}_3\phi = \phi\}, \qquad \mathcal{C}_2^s := \{\phi : \mathcal{P}_4\phi = -\phi, \mathcal{P}_3\phi = \phi\},$$
$$\mathcal{C}_3^s := \{\phi : \mathcal{P}_4\phi = \phi, \mathcal{P}_3\phi = -\phi\}.$$

Observe that, in particular, if ϕ_1 and ϕ_2 are two scalar functions such that $\phi_1 \in \mathcal{C}_1^s$ and $\phi_2 \in \mathcal{C}_m^s$ where $m = 2, 3$, then,

$$\int_{\Omega} \phi_1 \phi_2 = 0. \tag{11}$$

Definition 4 (*Symmetry Class for Vector Fields*). Suppose $\mathbf{w} = (w_1, w_2, w_3)$ is a vector field; then we define the following classes (see [7]):

$$\begin{aligned}
\mathcal{C}_1^v &:= \{\mathbf{w} : w_1 = \mathcal{P}_1 w_1 = \mathcal{P}_2 w_1 = \mathcal{P}_3 w_1, w_2 = -\mathcal{P}_1 w_2 = -\mathcal{P}_2 w_2 = \mathcal{P}_3 w_2,\\
&\qquad w_3 = -\mathcal{P}_1 w_3 = \mathcal{P}_2 w_3 = -\mathcal{P}_3 w_3\}\\
\mathcal{C}_2^v &:= \{\mathbf{w} : w_1 = -\mathcal{P}_1 w_1 = \mathcal{P}_2 w_1 = \mathcal{P}_3 w_1, w_2 = -\mathcal{P}_1 w_2 = \mathcal{P}_2 w_2 = \mathcal{P}_3 w_2,\\
&\qquad w_3 = \mathcal{P}_1 w_3 = -\mathcal{P}_3 w_3 = -\mathcal{P}_3 w_3\}\\
\mathcal{C}_3^v &:= \{\mathbf{w} : w_1 = -\mathcal{P}_1 w_1 = -\mathcal{P}_2 w_1 = -\mathcal{P}_3 w_1, w_2 = \mathcal{P}_1 w_2 = \mathcal{P}_2 w_2 = -\mathcal{P}_3 w_2,\\
&\qquad w_3 = \mathcal{P}_1 w_3 = -\mathcal{P}_2 w_3 = \mathcal{P}_3 w_3\}\\
\mathcal{C}_4^v &:= \{\mathbf{w} : w_1 = \mathcal{P}_1 w_1 = \mathcal{P}_2 w_1 = -\mathcal{P}_3 w_1, w_2 = -\mathcal{P}_1 w_2 = -\mathcal{P}_2 w_2 = -\mathcal{P}_3 w_2,\\
&\qquad w_3 = -\mathcal{P}_1 w_3 = \mathcal{P}_2 w_3 = \mathcal{P}_3 w_3\}\\
\mathcal{C}_5^v &:= \{\mathbf{w} : w_1 = \mathcal{P}_1 w_1 = -\mathcal{P}_2 w_1 = \mathcal{P}_3 w_1, w_2 = -\mathcal{P}_1 w_2 = \mathcal{P}_2 w_2 = \mathcal{P}_3 w_2,\\
&\qquad w_3 = -\mathcal{P}_1 w_3 = -\mathcal{P}_2 w_3 = -\mathcal{P}_3 w_3\}.
\end{aligned}$$

Writing $u_S = \xi \mathbf{h}^{(i)} + \xi_2 \mathbf{h}^{(2)}$ and using the symmetry definitions, we have the following results:

1. The Auxiliary fields possess the following symmetries [15,6,7]:

$$\begin{aligned}
&\mathbf{h}^{(1)} \in \mathcal{C}_1^v, \qquad \mathbf{h}^{(2)} \in \mathcal{C}_2^v\\
&\mathbf{H}^{(1)} \in \mathcal{C}_3^v, \qquad \mathbf{H}^{(2)} \in \mathcal{C}_4^v, \qquad \mathbf{H}^{(3)} \in \mathcal{C}_5^v.
\end{aligned} \tag{12}$$

2. Using Definitions 2 and 3 and Eqs. (8) and (9), we have that the inertial components of the torque are

$$\begin{aligned}
&\mathcal{M}_1^I(u_S) = \mathcal{M}_2^I(u_S) = 0\\
&\mathcal{M}_3^I = -\mathrm{Re}\,\xi_1\xi_2 \int_\Omega (\mathbf{h}^{(1)} \cdot \nabla \mathbf{h}^{(2)} + \mathbf{h}^{(2)} \cdot \nabla \mathbf{h}^{(1)}) \cdot \mathbf{H}^{(3)} = \mathrm{Re}\,\xi_1\xi_2 \mathcal{G}_I
\end{aligned}$$

where $\mathcal{G}_I$ is the integral quantity which we shall refer to as the torque coefficient. This last quantity has been calculated completely for varying eccentricities of a settling ellipsoid in [6]. We see that the values of the torque coefficient are zero for the eccentricities zero and one (i.e. for a sphere and a needle respectively), peaking at around $e = 0.9$.

3. The components of the shear-thinning or shear-thickening torque, employing the symmetries above along with Eq. (10), are

$$\mathcal{M}_i^{PL}(u_S) = -\int_\Omega [\mathbf{D}(\xi_1 \mathbf{h}^{(1)} + \xi_2 \mathbf{h}^{(2)}) : \mathbf{D}(\xi_1 \mathbf{h}^{(1)} + \xi_2 \mathbf{h}^{(2)})]^{\frac{n-1}{2}} \mathbf{D}(\xi_1 \mathbf{h}^{(1)} + \xi_2 \mathbf{h}^{(2)}) : \mathbf{D}(\mathbf{H}^{(i)}) \tag{13}$$

for $i = 1, 2, 3$. Therefore, using Definitions 1 and 2, we observe that

$$\mathbf{D}(\mathbf{h}^{(1)}) : \mathbf{D}(\mathbf{h}^{(1)}) \in \mathcal{C}_1^s, \qquad \mathbf{D}(\mathbf{h}^{(2)}) : \mathbf{D}(\mathbf{h}^{(2)}) \in \mathcal{C}_1^s, \qquad \mathbf{D}(\mathbf{h}^{(1)}) : \mathbf{D}(\mathbf{h}^{(2)}) \in \mathcal{C}_1^s \tag{14}$$

and therefore,

$$\begin{aligned}
&\mathbf{D}(\xi_1 \mathbf{h}^{(1)} + \xi_2 \mathbf{h}^{(2)}) : \mathbf{D}(\xi_1 \mathbf{h}^{(1)} + \xi_2 \mathbf{h}^{(2)}) \in \mathcal{C}_1^s\\
&\qquad \Rightarrow [\mathbf{D}(\xi_1 \mathbf{h}^{(1)} + \xi_2 \mathbf{h}^{(2)}) : \mathbf{D}(\xi_1 \mathbf{h}^{(1)} + \xi_2 \mathbf{h}^{(2)})]^{\frac{n-1}{2}} \in \mathcal{C}_1^s
\end{aligned} \tag{15}$$

for any value of the power, $\frac{n-1}{2}$. Furthermore, it is easily verified that

$$\xi_1 \mathbf{D}(\mathbf{h}^{(1)}) : \mathbf{D}(\mathbf{H}^{(1)}) + \xi_2 \mathbf{D}(\mathbf{h}^{(2)}) : \mathbf{D}(\mathbf{H}^{(1)}) \in \mathcal{C}_3^s,$$
$$\xi_1 \mathbf{D}(\mathbf{h}^{(1)}) : \mathbf{D}(\mathbf{H}^{(2)}) + \xi_2 \mathbf{D}(\mathbf{h}^{(2)}) : \mathbf{D}(\mathbf{H}^{(2)}) \in \mathcal{C}_3^s,$$
$$\xi_1 \mathbf{D}(\mathbf{h}^{(1)}) : \mathbf{D}(\mathbf{H}^{(3)}) + \xi_2 \mathbf{D}(\mathbf{h}^{(2)}) : \mathbf{D}(\mathbf{H}^{(3)}) \in \mathcal{C}_2^s.$$

Accounting for these symmetries in Eq. (13) and employing Eq. (11), it follows that the components of the shear-thinning torque

$$\mathcal{M}_i^{PL}(u_S) = 0 \tag{16}$$

for each $i = 1, 2, 3$. Therefore, the shear-thinning effects contribute nothing towards the torque, at low *Re*. Note that the argument stated above is independent of the choice of the power $\frac{n-1}{2}$ and therefore applies equally to shear-thickening liquids.

Hence, in conclusion, the net non-zero torque acting on the body $\mathcal{B}$ is given by

$$\mathcal{M}_3 = -\mathrm{Re}\, \xi_1 \xi_2 \mathcal{G}_I = -\mathrm{Re}\, |\xi|^2 \mathcal{G}_I \sin(\theta) \cos(\theta) \tag{17}$$

where we choose, without loss of generality, $\xi = (\xi_1, \xi_2, 0)$ which we further decompose into polar coordinates, with θ measuring the angle between ξ and the horizontal axis. It is seen from Eq. (17) that for the net torque to vanish, $\theta = 0$ or 90 degrees, just as in the case of a Newtonian fluid. Our results seem to indicate that pure shear-thinning or shear-thickening effects play no role in causing the tilt angle, at very low Reynolds numbers. Eq. (13) combined with the Eq. (17) tells us that in a power-law fluid, the surviving torque is due to inertial effects alone. Hence an ellipsoid, sedimenting in a liquid which can be modeled by the power-law fluid equations, will orient itself with its major axis either parallel or perpendicular to the direction of gravity with the former state being the stable one. Therefore, at first order in *Re* and *We, independent competing effects of inertia, normal stresses and shear-thinning do not explain the tilt angle*. Consequently, the last possibility one is left to explore is a fluid model that couples shear-thinning with normal stress effects, which will be the subject of a later paper.

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